

Projective Closure and Cylindrical Representation of the Dual Number Plane through Möbius Action

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Abstract. In this paper, we have studied the Möbius action of $SL(2; \mathbb{R})$ on the dual number plane. Dual numbers are of the form $a + \varepsilon b$, where $\varepsilon^2 = 0$. In analogy with the compactification of the complex plane by the Riemann sphere, we have shown that the dual number plane can be represented in a cylindrical form. We have classified $SL(2; \mathbb{R})$ into the three one-parameter subgroups A , N and K using Iwasawa decomposition, and using these subgroups, we have represented the dual number plane as a homogeneous space of $SL(2; \mathbb{R})$. We have then showed that this space can be identified with $S^1 \times \mathbb{R}^+$, which has the form of a cylinder. By introducing homogeneous coordinates and a line at infinity, we have obtained the projective closure of the dual number plane. We further showed that the Möbius action of $PSL(2; \mathbb{R})$ gives a bijective correspondence between the dual number plane and this cylindrical representation. This construction provides a simple geometric model for the compactification of the dual number plane and extends the familiar idea of the Riemann sphere to the parabolic case of hypercomplex numbers.

Keywords. dual numbers, cylinder, homogeneous coordinates, projective space, Möbius action, projective special linear group, compactification.

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1 Introduction

The study of hypercomplex numbers is an intriguing and multifaceted area of mathematics, characterized by its exploration of various extensions and generalizations of the complex number system. Brewer [7] made a significant contribution to this field by stating that, up to isomorphism, there are three kinds of hypercomplex numbers: complex, dual, and double numbers. These types are respectively termed as elliptic, parabolic, and hyperbolic cases (EPH cases) in the extensive works of Kisil [11–13]. Each of these cases represents a unique algebraic structure with distinct geometric interpretations and properties.

Complex numbers, which are perhaps the most familiar of the three, can be compactified through the Riemann sphere by adding a point at infinity [16]. This process involves extending the complex plane to include this additional point, resulting in a closed and compact space known as the extended complex plane or the Riemann sphere. This extended plane allows for the application of the Möbius transformation, a set of conformal mappings that act as bijections between the original complex plane and its extended version. These transformations preserve the geometric properties of figures and are fundamental in complex analysis and other areas of mathematics.

In our research, we seek to extend this concept of compactification to dual numbers, another type of hypercomplex number. Dual numbers are defined as expressions of the form $a + \varepsilon b$, where a and b are real numbers, and ε is a nilpotent element with the property $\varepsilon^2 = 0$. The dual number plane can be thought of as a two-dimensional space similar to the complex

plane but with different algebraic rules. Our goal is to extend the dual number plane in a manner that compactifies it as a cylinder. This involves defining new coordinates and structures that can encompass the entire extended plane, including the “point at infinity” analogous to the complex case.

Kisil’s works have been instrumental in exploring the algebraic and geometrical aspects of the Möbius action of $SL(2; \mathbb{R})$ [15] on the EPH cases. The Möbius transformations in this context can be seen as actions of the special linear group $SL(2; \mathbb{R})$, a group of 2×2 real matrices with determinant 1. This group acts on the hypercomplex numbers in ways that preserve certain geometric properties [1, 3, 6]. Kisil utilized the Erlangen Program of Felix Klein [14], which provides a framework for understanding geometry as the study of properties invariant under a group of transformations. This program is crucial for analyzing the symmetries and invariants in the structures formed by hypercomplex numbers.

To delve deeper into the structure of $SL(2; \mathbb{R})$, we employ the Iwasawa decomposition, a technique that decomposes the group into three one-parameter subgroups A , N and K [2, 10]. This decomposition allows for a detailed analysis of the group’s structure and its actions. By representing dual numbers as a homogeneous space of $SL(2; \mathbb{R})$, we can explore their geometric properties in a more structured way. Specifically, we express a cylinder as a product of $\mathbb{S}^1 \times \mathbb{R}^+$, where $\mathbb{S}^1$ represents the unit circle and $\mathbb{R}^+$ represents the positive real numbers. This product space provides a natural setting for compactifying the dual number plane.

We introduce homogeneous coordinates for the extended dual number plane and define a line at infinity that wraps around into a cylinder, thus achieving the desired compactification. This approach parallels the compactification of the complex plane via the Riemann sphere but adapts it to the unique algebraic structure of dual numbers. Furthermore, by extending $SL(2; \mathbb{R})$ into projective space [8, 17, 18], we define the projective special linear group $PSL(2; \mathbb{R})$ [7]. The projective group $PSL(2; \mathbb{R})$ consists of equivalence classes of matrices in $SL(2; \mathbb{R})$ under scalar multiplication, and its Möbius action on the extended dual number plane is bijective, preserving the plane’s structure and properties.

Our work builds upon the foundational work of Brewer and Kisil, applying sophisticated concepts from the Erlangen Program and the Iwasawa decomposition to explore the compactification and transformation properties of dual numbers. By extending the dual number plane into a cylindrical form and leveraging the Möbius action of $PSL(2; \mathbb{R})$, we aim to deepen the understanding of the algebraic and geometric properties of hypercomplex numbers. This work not only extends classical results in complex analysis to the realm of dual numbers but also provides new insights into the rich interplay between algebra, geometry, and group theory in the context of hypercomplex systems.

1.1 Preliminaries

We first discuss certain important terminologies related to our work.

Definition 1.1. (Group action) A group action of a group $\mathbb{G}$ on a set $\mathbb{X}$ is a map from $\mathbb{G} \times \mathbb{X}$ to $\mathbb{X}$ (written as $g \cdot x$ for all $g \in \mathbb{G}$, $x \in \mathbb{X}$) such that, [9]

- (i) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in \mathbb{G}$, $x \in \mathbb{X}$ and
- (ii) $e \cdot x = x$ for all $x \in \mathbb{X}$ (e is the identity element of $\mathbb{G}$).

This definition encapsulates the idea that the group G acts on the set X in a way that respects the group operation, preserving the structure of X under the action of G . Group

extensively in various branches of mathematics such as algebra and geometry.

Definition 1.2. (*Stabilizer*) The stabilizer of an element $h \in \mathbb{H}$ under a group action from $\mathbb{G}$ is the set,

$$\text{Stab}(h) = \{g \in \mathbb{G} \mid g \cdot h = h\}.$$

This set consists of all elements in $\mathbb{G}$ that leave the element $h \in \mathbb{H}$ unchanged under the group action. The concept of the stabilizer helps in understanding the symmetry properties of elements under group actions, providing insight into the structure of the group $\mathbb{G}$ and its action on the set $\mathbb{H}$. In many cases, the stabilizer can reveal important invariants and symmetries that are crucial in the study of group theory and its applications.

Definition 1.3. (*Dual numbers*) The numbers of the form $a + b\varepsilon$, where $\varepsilon^2 = 0$, $a, b \in \mathbb{R}$ are called dual numbers which are represented by the symbol $\mathbb{R}^p$, where ε is the dual unit.

Dual numbers extend the real numbers by introducing a new element ε that squares to zero, creating a new algebraic structure. This extension is useful in various applications, such as automatic differentiation, where the dual unit ε can be used to represent infinitesimal quantities. The algebra of dual numbers shares similarities with complex numbers, but with distinct properties that make them valuable in different contexts, particularly in geometric and algebraic studies.

Definition 1.4. (*Iwasawa decomposition of $SL(2; \mathbb{R})$*) The Iwasawa decomposition of $SL(2; \mathbb{R})$ shows that it can be uniquely decomposed in the form $SL(2; \mathbb{R}) = ANK$, where

(a) $A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha (> 0) \in \mathbb{R} \right\}$ is the simply connected abelian group of 2×2 matrices.

(b) $N = \left\{ \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} : \nu \in \mathbb{R} \right\}$ is the group of 2×2 nilpotent matrices.

(c) $K = \left\{ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} : \phi \in \mathbb{R} \right\}$ is the maximal compact group, contains the centre of $SL(2; \mathbb{R})$, and is normal in A and N .

This decomposition plays a critical role in the analysis of Lie groups and their representations, providing a structured way to understand the composition and properties of the special linear group $SL(2; \mathbb{R})$.

Definition 1.5. We define Möbius transformation as a function $T : \mathbb{R}^p \rightarrow \mathbb{R}^p$ which can also be represented as an $SL(2; \mathbb{R})$ -action on $\mathbb{R}^p$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az + b}{cz + d} = T(z) = w; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{R}), \quad z \in \mathbb{R}^p.$$

This action is fundamental as it preserves the algebraic and geometric structures of the dual number plane, enabling various analyses and manipulations within this extended space.

Definition 1.6. (*Homogeneous Space*) A topological space X together with an abstract group $(G, \cdot)$, which acts on X transitively is said to be a homogeneous space [12].

that $g \cdot x = y$. This definition implies that the group action is such that the group G can move any point in the space X to any other point, demonstrating a high degree of symmetry. Homogeneous spaces are essential in geometry and physics, as they provide models for spaces with uniform properties and symmetries. Examples of homogeneous spaces include Euclidean spaces, spheres, and projective spaces, each exhibiting different types of symmetry and group actions.

2 Representation of a cylinder as a Cartesian product

Representing the cylinder as $\mathbb{S}^1 \times \mathbb{R}^+$ is a sophisticated geometric and algebraic endeavor, one that brings together fundamental concepts from topology and real analysis to form a coherent and comprehensive model. The set $\mathbb{S}^1$ denotes all points on a unit circle, which in Cartesian coordinates can be expressed as $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. This circle is a simple yet profound object in mathematics, representing a closed curve where each point is equidistant from a central point. Alternatively, the unit circle can be parameterized using an angle θ that ranges from 0 to 2π . In this form, it is written as $\mathbb{S}^1 = \{e^{i\theta} : \theta \in [0, 2\pi]\}$, where $e^{i\theta} = \cos \theta + i \sin \theta$ represents points on the circle in the complex plane. This parameterization captures the rotational symmetry of the circle, making it a versatile tool for various mathematical analysis. On the other hand, $\mathbb{R}^+$ signifies the set of positive real numbers, which can be denoted as the interval $(0, \infty)$. This set extends infinitely in the positive direction on the real number line, representing values that increase without bound. The importance of $\mathbb{R}^+$ lies in its unbounded nature, which, when combined with $\mathbb{S}^1$, results in an infinite extension along one axis.

The Cartesian product $\mathbb{S}^1 \times \mathbb{R}^+$ combines these two sets, producing a space where each point is a pair (θ, r) . Here, θ varies over the interval $[0, 2\pi]$, representing an angle around the circle, while r varies over $\mathbb{R}^+$, representing a radial distance from the origin along the vertical axis. This combination effectively describes a cylindrical surface, where each value of r corresponds to a different circle of radius r , and these circles are stacked vertically to create an infinitely long cylinder. Visualizing $\mathbb{S}^1 \times \mathbb{R}^+$ as a cylinder involves imagining a circle for each value of r in $\mathbb{R}^+$, where r represents the radius or the distance from the origin. A point in this cylinder can be parameterized as $(r \cos \theta, r \sin \theta)$, where $\theta \in [0, 2\pi]$ determines the position around the circle, and $r \in \mathbb{R}^+$ determines the distance from the center of the circle along the vertical axis. Thus, for each fixed r , the points $(r \cos \theta, r \sin \theta)$ trace out a circle, and as r varies, these circles stack to form an infinitely long cylindrical surface.

Overall, this construction allows us to see $\mathbb{S}^1$ as a circle and $\mathbb{R}^+$ as a line stretching infinitely in the positive direction. When combined in $\mathbb{S}^1 \times \mathbb{R}^+$, they create a cylindrical shape with an infinite extent. Each point on this cylinder can be uniquely identified by its angle θ and its height r , creating a continuous and seamless cylindrical surface that extends upwards without bound. This perspective is crucial for understanding the compactification of the dual number plane, where we aim to encapsulate the entire plane, including the points at infinity, within a compact cylindrical form.

Theorem 3.1. *The stabilizer of ε under Möbius action is the subgroup N' of matrices given by the set $N' = \left\{ \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} : \nu \in \mathbb{R} \right\}$ [12].*

Proof. To prove that N' is the stabilizer of ε under the Möbius action, we start by considering the Möbius transformation action of the matrix $\begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}$ from the group N' on the dual unit ε . The Möbius transformation for an element ε under this matrix action is given by,

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \cdot \varepsilon &\mapsto \frac{\varepsilon 1 + 0}{\varepsilon \nu + 1} \\ &= \frac{\varepsilon}{1 + \varepsilon \nu} \left(\frac{1 - \varepsilon \nu}{1 - \varepsilon \nu} \right) \\ &= \frac{\varepsilon - \varepsilon^2 \nu}{1 - \varepsilon^2 \nu^2} \\ &= \varepsilon. \quad (\because \varepsilon^2 = 0) \end{aligned}$$

Therefore, we have shown that the Möbius action of $\begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}$ on ε results in ε itself. Hence, N' is the stabilizer of ε , as it leaves ε unchanged under its group action. $\square$

Theorem 3.2. *The stabilizer N' is conjugate to the subgroup N [12].*

Proof. Let $\begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \in N'$ and $\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \in N$. Now, we have the matrix $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2; \mathbb{R})$ such that

$$\begin{aligned} &P \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} P^{-1} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\nu & 0 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ \nu & 0 \end{pmatrix}. \quad (\because \nu \text{ is just a notation}) \end{aligned}$$

Hence, N' is conjugate to N . $\square$

Thus, the parabolic upper half-plane is a model for the homogeneous space $SL(2; \mathbb{R})/N$ and we can express the space $\mathbb{R}^p$ as

$$\begin{aligned} &SL(2; \mathbb{R})/N \\ &= (KAN)/N \\ &= KA \\ &= K \times A \\ &= \left\{ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}; \phi \in \mathbb{R} \right\} \times \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}; \alpha(>0) \in \mathbb{R} \right\} \\ &\cong \mathbb{S}^1 \times \mathbb{R}^+. \end{aligned}$$

Theorem 3.3. The map $f : \mathbb{R}^p \rightarrow \mathbb{S}^1 \times \mathbb{R}^+$ given by $f(a + \varepsilon b) = (e^{ib}, e^a)$ where $a, b \in \mathbb{R}$ is a bijection.

Proof. For the bijection to $\mathbb{S}^1$, we map b to e^{ib} . This maps b to a point on the unit circle in the complex plane ensuring $|e^{ib}| = 1$. The real part a of the dual number corresponds to the positive real axis $\mathbb{R}^+$. We map a to e^a , which ensures $e^a > 0$ for all $a \in \mathbb{R}$.

(a) Injectivity: Suppose $(e^{ib}, e^a) = (e^{ib'}, e^{a'})$ for some $a + \varepsilon b, a' + \varepsilon b' \in \mathbb{R}^p$. Then,

$$\begin{aligned} e^{ib} &= e^{ib'} \\ \implies b &= b' + 2k\pi \text{ for some integer } k \\ \implies b &\equiv b' \pmod{2\pi}. \end{aligned}$$

Since b and b' are in the range $[0, 2\pi]$, we have $b = b'$. Again,

$$\begin{aligned} e^a &= e^{a'} \\ \implies e^{a-a'} &= e^0 \\ \implies a - a' &= 0 \\ \implies a &= a'. \end{aligned}$$

Hence, $f(a + \varepsilon b) = f(a' + \varepsilon b') \implies a + \varepsilon b = a' + \varepsilon b'$.

(b) Surjectivity: For any $(e^{i\theta}, r) \in \mathbb{S}^1 \times \mathbb{R}^+$, we can find $a = \ln(r)$ and $b = \theta$. Then, $e^a = r$ and $e^{ib} = e^{i\theta}$, so $a + \varepsilon b = \ln(r) + \varepsilon\theta$.

Therefore, the mapping $f(a + \varepsilon b) = (e^{ib}, e^a)$ establishes a bijection from dual numbers to $\mathbb{S}^1 \times \mathbb{R}^+$. Each element in $\mathbb{S}^1 \times \mathbb{R}^+$ corresponds uniquely to a dual number, and vice versa ensuring the bijective nature of the mapping. $\square$

4 The plane of dual numbers and its projective closure

For the complex numbers, the Riemann Sphere is constructed by adding a point at infinity to the complex plane, effectively creating a closed, compact space that can be visualized as a sphere. In the realm of projective geometry, this transformation is captured by the extended complex plane, denoted as $\hat{\mathbb{C}}$, which is formed by appending this point at infinity, so that $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This construction allows us to treat the complex plane as a sphere where every point in the plane maps onto the surface of the sphere, with the point at infinity serving as the north pole. The elegance of the Riemann Sphere lies in its ability to provide a comprehensive framework for understanding complex functions and their behavior at infinity, making it a powerful tool in complex analysis.

To gain a deeper understanding of the closure of the dual number plane, we draw an analogy to the extended complex plane and the concept of the Riemann Sphere. The projective closure of the dual number plane involves adding a "line at infinity", denoted by l_∞ . This line at infinity plays a role similar to the point at infinity on the Riemann Sphere, serving to compactify the dual number plane and create a closed, finite space. In this context, we consider the projective dual numbers $\mathbb{P}^1(\mathbb{R}^p)$ as the dual number plane augmented with this line at infinity. By incorporating l_∞ , we create a geometric transformation that results in a cylindrical surface, where the dual number plane $\mathbb{R}^p$ is effectively wrapped around this cylinder.

To explore this geometric transformation, we begin by representing the dual number plane $\mathbb{R}^p$ with points (x, y) , where x and y are real numbers. For simplicity, we consider the standard Cartesian plane, where each point in the plane is represented using homogeneous coordinates (X, Y, Z) . In projective geometry, these homogeneous coordinates are used to describe points such that $(x, y) = \left(\frac{X}{Z}, \frac{Y}{Z}\right)$ for $Z \neq 0$. Points at infinity in this framework are those where $Z = 0$. The line at infinity l_∞ consists of all points with coordinates $(X, Y, 0)$, representing directions rather than locations.

Visualizing the Cartesian plane as lying flat in a three-dimensional space, we can imagine that each line in this plane extends infinitely in both directions. The collection of all these directions forms the line at infinity l_∞ . When we incorporate this line at infinity into the plane, we effectively compactify it, transforming it into a closed and finite space. This compactification can be visualized by wrapping the plane into a cylindrical surface. Imagine taking the edges of the plane and rolling it into a tube, so that the top and bottom edges, which originally extended to infinity, now meet to form a continuous loop.

This process transforms the infinite expanse of the plane into a finite cylindrical surface, where the dual number plane is wrapped around the cylinder. Each point on the plane corresponds to a point on the cylindrical surface, and the lines that extend to infinity now loop around the cylinder. The line at infinity l_∞ , thus becomes an integral part of the cylindrical surface, ensuring that the plane is closed and compact.

The construction of $\mathbb{S}^1 \times \mathbb{R}^+$ combines the simplicity of the circle with the infinite nature of the positive real line to form a cylindrical surface that extends indefinitely. This cylindrical surface, when compactified with the line at infinity, provides a natural setting for exploring the geometric and algebraic properties of dual numbers. The circular cross-section $\mathbb{S}^1$ ensures rotational symmetry, while the infinite extension of $\mathbb{R}^+$ accommodates the unbounded nature of dual numbers.

To delve deeper into the mathematical representation of this compactification, we consider points in the dual number plane as pairs (x, y) . In homogeneous coordinates, these points are represented as (X, Y, Z) , with the relationship $(x, y) = \left(\frac{X}{Z}, \frac{Y}{Z}\right)$ for $Z \neq 0$. Points at infinity, where $Z = 0$, form the line at infinity l_∞ . By adding this line at infinity to the plane, we compactify it into a cylindrical surface. This can be visualized by imagining the Cartesian plane being wrapped into a cylinder, with the line at infinity forming the seam of the cylinder.

Hence, the projective closure of the dual number plane through the addition of a line at infinity results in a cylindrical surface. This transformation mirrors the extension of the complex plane to the Riemann Sphere, providing a compact and closed space that facilitates the study of dual numbers and their geometric properties. By visualizing this process, we gain a deeper understanding of the interplay between algebraic structures and geometric transformations in the realm of projective geometry. The resulting cylinder serves as a powerful tool for exploring the properties of dual numbers, their compactification, and the transformations that preserve their structure.

5 Möbius transformation in a projective space

To thoroughly understand the concept of projective space in the context of dual numbers, we start by defining equivalence relations and their corresponding equivalence classes.

Definition 5.1. Given, $z = (x, y)$, $w = (u, v) \in \mathbb{R}^p$. Then $z \sim w$ if and only if there exists a non-zero scalar λ such that $(x, y) = (\lambda u, \lambda v)$. Then, the projective space of dual numbers is defined to be the set of equivalence classes of the equivalence relation ' $\sim$ ' i.e., $[\gamma]$

$$\mathbb{P}^1(\mathbb{R}^p) = \{[z, w]; z, w \in \mathbb{R}^p \text{ and } z \sim w\}.$$

To map elements from the dual number plane $\mathbb{R}^p$ to this projective space, we define a map $\tau : \mathbb{R}^p \rightarrow \mathbb{P}^1(\mathbb{R}^p)$ by $\tau(z) = [z, 1]$ where $z = (x, y) \in \mathbb{R}^p$ is represented in the form of a fraction $z = \frac{x}{y}$, provided $y \neq 0$. This mapping allows us to handle dual numbers in a projective context, making it easier to perform operations and transformations that require a notion of infinity or compactification. In a projective space, the Möbius action can be represented by $\tau(w) = [w, 1]$, which maintains the integrity of the projective transformation.

Definition 5.2. The Special Linear group is defined as,

$$SL(2; \mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \det(A) = 1 \right\}.$$

The Projective Special Linear group is the quotient group

$$PSL(2; \mathbb{R}) = SL(2; \mathbb{R}) / \{\lambda I; \lambda \neq 0\}.$$

Then, $PSL(2; \mathbb{R})$ -action on $\mathbb{P}^1(\mathbb{R}^p)$ can be defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [z, 1] \mapsto \left[\frac{az + b}{cz + d}, 1 \right].$$

We now use the Projective Special Linear group to establish the projective Möbius transformation between $\mathbb{R}^p$ and its corresponding projective space which closes up as a cylinder. This is represented by the $PSL(2; \mathbb{R})$ -action on $\mathbb{P}^1(\mathbb{R}^p)$.

Theorem 5.1. The projective Möbius transformation given by $\tau \circ T : \mathbb{R}^p \rightarrow \mathbb{P}^1(\mathbb{R}^p)$ can be represented by the $PSL(2; \mathbb{R})$ -action on $\mathbb{P}^1(\mathbb{R}^p)$.

Proof. Given a Möbius transformation $T(z)$, where $z \in \mathbb{R}^p$ we have

$$\begin{aligned} (\tau \circ T)(z) &= \tau(T(z)) \\ &= \tau\left(\frac{az + b}{cz + d}\right) \\ &= \left[\frac{az + b}{cz + d}, 1 \right], \text{ which is the } PSL(2; \mathbb{R})\text{-action on } \mathbb{P}^1(\mathbb{R}^p). \end{aligned}$$

This map is also bijective in nature, since the condition $ad - bc = 1 \neq 0$ ensures bijectivity. $\square$

By defining equivalence classes in the dual number plane and mapping them to a projective space, we establish a framework for handling dual numbers in a projective context. The Möbius transformations facilitated by $SL(2; \mathbb{R})$ and extended to $PSL(2; \mathbb{R})$ provide powerful tools for compactifying the dual number plane and performing sophisticated mathematical operations. This approach not only extends classical results in complex analysis to dual numbers but also enriches our understanding of hypercomplex number systems and their geometric properties.

6 Conclusion

In this paper, we studied the Möbius action of $SL(2; \mathbb{R})$ on the dual number plane. Using the Iwasawa decomposition $SL(2; \mathbb{R}) = ANK$, we represented the dual number plane as a homogeneous space and identified it with $S^1 \times \mathbb{R}^+$, giving it a cylindrical representation. We then introduced homogeneous coordinates and a line at infinity to obtain the projective closure of the dual number plane.

We further showed that the Möbius action of $PSL(2; \mathbb{R})$ gives a bijective correspondence between the dual number plane and its cylindrical representation. Thus, our construction provides a geometric compactification of the dual number plane, analogous to the Riemann sphere for the complex plane. This work also demonstrates the connection between dual numbers, projective geometry, and the group action of $SL(2; \mathbb{R})$.

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